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# A Twisted Generalization of Lie-Yamaguti Algebras

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## Abstract

A twisted generalization of Lie-Yamaguti algebras, called Hom-Lie-Yamaguti algebras, is defined. Hom-Lie-Yamaguti algebras generalize multiplicative Hom-Lie triple systems (and subsequently ternary multiplicative Hom-Nambu algebras) and Hom-Lie algebras in the same way as Lie-Yamaguti algebras generalize Lie triple systems and Lie algebras. It is shown that the category of (multiplicative) Hom-Lie-Yamaguti algebras is closed under twisting by self-morphisms. Constructions of Hom-Lie-Yamaguti algebras from ordinary Lie-Yamaguti algebras and Malcev algebras are given. Using the well-known classification of real two-dimensional Lie-Yamaguti algebras, examples of real two-dimensional Hom-Lie-Yamaguti algebras are given.

**Mathematics Subject Classification:** 17A30, 17D99

**Keywords:** Lie-Yamaguti algebra (i.e. generalized Lie triple system, Lie triple algebra), Malcev algebra, Hom-Lie algebra, Hom-Lie triple system, Hom-Nambu algebra, Hom-Akivis algebra

# 1 Introduction

Using the Bianchi identities, K. Nomizu [15] characterized, by some identities involving the torsion and the curvature, reductive homogeneous spaces with some canonical connection. K. Yamaguti [19] gave an algebraic interpretation of these identities by considering the torsion and curvature tensors of Nomizu's canonical connection as a bilinear and a trilinear algebraic operations satisfying some axioms, and thus defined what he called a "general Lie triple system". M. Kikkawa [8] used the term "Lie triple algebra" for such an algebraic object. More recently, M.K. Kinyon and A. Weinstein [9] introduced the term "Lie-Yamaguti algebra" for this object.

A *Lie-Yamaguti algebra*  $(V, *, \{, \})$  is a vector space  $V$  together with a binary operation  $*$  :  $V \times V \rightarrow V$  and a ternary operation  $\{, \}$  :  $V \times V \times V \rightarrow V$  such that

$$(LY1) \quad x * y = -y * x,$$

$$(LY2) \quad \{x, y, z\} = -\{y, x, z\},$$

$$(LY3) \quad \circlearrowleft_{(x,y,z)} [(x * y) * z + \{x, y, z\}] = 0,$$

$$(LY4) \quad \circlearrowleft_{(x,y,z)} \{x * y, z, u\} = 0,$$

$$(LY5) \quad \{x, y, u * v\} = \{x, y, u\} * v + u * \{x, y, v\},$$

$$(LY6) \quad \{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} + \{u, \{x, y, v\}, w\} \\ + \{u, v, \{x, y, w\}\},$$

for all  $u, v, w, x, y, z$  in  $V$  and  $\circlearrowleft_{(x,y,z)}$  denotes the sum over cyclic permutation of  $x, y, z$ .

In [2] the notation "LY-algebra" is used for "Lie-Yamaguti algebra". So, likewise, below we will write "Hom-LY algebra" for "Hom-Lie-Yamaguti algebra".

Observe that if  $x * y = 0$ , for all  $x, y$  in  $V$ , then  $(V, *, \{, \})$  reduces to a *Lie triple system*  $(V, \{, \})$  as defined in [18]. From the other hand, if  $\{x, yz, \} = 0$  for all  $x, y, z$  in  $V$ , then  $(V, *, \{, \})$  is a Lie algebra  $(V, *)$ . Originally, N. Jacobson [7] defined a Lie triple system as a submodule of an associative algebra that is closed under the iterated commutator bracket.

In this paper we consider a Hom-type generalization of LY algebras that we call Hom-LY algebras. Roughly, a Hom-type generalization of a given type of algebras is defined by twisting the defining identities of that type of algebras by a self-map in such a way that, when the twisting map is the identity map, one recovers the original type of algebras. The systematic study of Hom-algebras was initiated by A. Makhlouf and S.D. Silvestrov [13], while D. Yau [23] gave a

general construction method of Hom-type algebras starting from usual algebras and a twisting self-map. For information on various types of Hom-algebras, one may refer to [1], [6], [11]-[13], [22]-[25].

A Hom-type generalization of  $n$ -ary Lie algebras,  $n$ -ary Nambu algebras and  $n$ -ary Nambu-Lie algebras (i.e. Filippov  $n$ -ary algebras) called  $n$ -ary Hom-Lie algebras,  $n$ -ary Hom-Nambu algebras and  $n$ -ary Hom-Nambu-Lie algebras respectively, is considered in [1]. Such a generalization is extended to the one of Hom-Lie triple systems and Hom-Jordan triple systems in [25]. We point out that the class of (multiplicative) Hom-LY algebras encompasses the ones of multiplicative ternary Hom-Nambu algebras, multiplicative Hom-Lie triple systems (hence Jordan and Lie triple systems), Hom-Lie algebras (hence Lie algebras) and LY algebras.

The rest of the paper is organized as follows. In section 2 some basic facts on Hom-algebras and  $n$ -ary Hom-algebras are recalled. The emphasis point here is that the definition of a Hom-triple system (Definition 2.3) is more restrictive than the D. Yau's in [25]. However, with this vision of a Hom-triple system, we point out that any non-Hom-associative algebra (i.e. nonassociative Hom-algebra or Hom-nonassociative algebra) has a natural structure of (multiplicative) Hom-triple system (this is the Hom-counterpart of a similar well-known result connecting nonassociative algebras and triple systems). Then we give the definition of a Hom-LY algebra and make some observations on its relationships with some types of ternary Hom-algebras and with LY algebras. In section 3 we show that the category of Hom-LY algebras is closed under twisting by self-morphisms (Theorem 3.1). Subsequently, we show a way to construct Hom-LY algebras from LY algebras (or Malcev algebras) by twisting along self-morphisms (Corollary 3.2 and Corollary 3.3); this is an extension to binary-ternary algebras of a result due to D. Yau ([23], Theorem 2.3. Such an extension is first mentioned in [6], Corollary 4.5). In section 4 examples of real two-dimensional Hom-LY algebras are constructed, relaying on the classification of (real) two-dimensional LY algebras [5], [21].

All vector spaces and algebras throughout are considered over a ground field  $\mathbb{K}$  of characteristic 0.

## 2 Ternary Hom-algebras. Definitions

We recall some basic facts about Hom-algebras, including ternary Hom-Nambu algebras. We note that the definition of a Hom-triple system given here (see Definition 2.3) is slightly more restrictive than the one given by D. Yau [25].

Then we give the definition of the main object of this paper (see Definition 2.6) and show its relationships with known structures such as ternary Hom-Nambu algebras, Hom-Lie triple systems, Hom-Lie algebras or Lie-Yamaguti algebras.

For definitions of  $n$ -ary Hom-algebras ( $n$ -ary Hom-Nambu and Hom-Nambu-Lie algebras,  $n$ -ary Hom-Lie algebras, etc.) we refer to [1], [25]. For information on origins of Nambu algebras, one may refer to [14], [17]. Here, for our purpose, we restrict our concern to ternary Hom-algebras. In fact, as we shall see below, a Hom-Lie-Yamaguti algebra is some multiplicative ternary Hom-Nambu algebra with an additional binary anticommutative operation satisfying some compatibility conditions.

**Definition 2.1.** ([25]). *A ternary Hom-algebra  $(V, [, ], \alpha = (\alpha_1, \alpha_2))$  consists of a  $\mathbb{K}$ -module  $V$ , a trilinear map  $[\cdot, \cdot, \cdot] : V \times V \times V \rightarrow V$ , and linear maps  $\alpha_i : V \rightarrow V$ ,  $i = 1, 2$ , called the **twisting maps**. The algebra  $(V, [, ], \alpha = (\alpha_1, \alpha_2))$  is said **multiplicative** if  $\alpha_1 = \alpha_2 := \alpha$  and  $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$  for all  $x, y, z \in V$ .*

For convenience, we assume throughout this paper that all Hom-algebras are multiplicative.

**Definition 2.2.** ([1]). *A (multiplicative) ternary Hom-Nambu algebra is a (multiplicative) ternary Hom-algebra  $(V, [, ], \alpha)$  satisfying*

$$\begin{aligned} [\alpha(x), \alpha(y), [u, v, w]] &= [[x, y, u], \alpha(v), \alpha(w)] + [\alpha(u), [x, y, v], \alpha(w)] \\ &\quad + [\alpha(u), \alpha(v), [x, y, w]], \end{aligned} \tag{2.1}$$

for all  $u, v, w, x, y \in V$ .

The condition (2.1) is called the *ternary Hom-Nambu identity*. In general, the ternary Hom-Nambu identity reads:

$$\begin{aligned} [\alpha_1(x), \alpha_2(y), [u, v, w]] &= [[x, y, u], \alpha_1(v), \alpha_2(w)] + [\alpha_1(u), [x, y, v], \alpha_2(w)] \\ &\quad + [\alpha_1(u), \alpha_2(v), [x, y, w]], \end{aligned}$$

for all  $u, v, w, x, y \in V$ , where  $\alpha_1$  and  $\alpha_2$  are linear self-maps of  $V$ .

**Definition 2.3.** *A (multiplicative) Hom-triple system is a (multiplicative) ternary Hom-algebra  $(V, [, ], \alpha)$  such that*

$$(i) \quad [u, v, w] = -[v, u, w],$$

(ii)  $\circlearrowleft_{(u,v,w)}[u, v, w] = 0,$   
 for all  $u, v, w \in V.$

**Remark.** A more general definition of a Hom-triple system is given by D. Yau [25] without the requirements (i), (ii) as in Definition 2.3 above. Our definition here is motivated by the concern of giving a Hom-type analogue of the relationships between nonassociative algebras and triple systems (see Remark below after Proposition 2.4).

A Hom-algebra in which the Hom-associativity is not assumed is called a nonassociative Hom-algebra [12] or a Hom-nonassociative algebra [22] (the expression of “non-Hom-associative” Hom-algebra is used in [6] for that type of Hom-algebras). With the notion of a Hom-triple system as above, we have the following

**Proposition 2.4.** Any non-Hom-associative Hom-algebra is a Hom-triple system.

**Proof.** Let  $(A, \cdot, \alpha)$  be a non-Hom-associative algebra. Then  $(A, [, ], as(, ,), \alpha)$  is a Hom-Akivis algebra with respect to  $[x, y] := x \cdot y - y \cdot x$  (commutator) and  $as(x, y, z) := xy \cdot \alpha(z) - \alpha(x) \cdot yz$  (Hom-associator), i.e. the Hom-Akivis identity

$$\circlearrowleft_{(x,y,z)}[[x, y], \alpha(z)] = \circlearrowleft_{(x,y,z)}as(x, y, z) - \circlearrowleft_{(x,y,z)}as(y, x, z)$$

holds for all  $x, y, z$  in  $A$  ([6]). Now define

$$[x, y, z] := [[x, y], \alpha(z)] - as(x, y, z) + as(y, x, z)$$

for all  $x, y, z$  in  $A$ . Then  $[x, y, z] = -[y, x, z]$  and the Hom-Akivis identity implies that  $\circlearrowleft_{(x,y,z)}[x, y, z] = 0$ . Thus  $(A, [, ], \alpha)$  is a Hom-triple system.  $\square$

**Remark.** For  $\alpha = Id$  (the identity map), we recover the triple system with ternary operation  $[[x, y], z] - (x, y, z) + (y, x, z)$  that is associated to each nonassociative algebra, since any nonassociative algebra has a natural Akivis algebra structure with respect to the commutator and associator operations  $[x, y]$  and  $(x, y, z)$ , for all  $x, y, z$  (see, e.g., remarks and references in [6]).

**Definition 2.5.** ([25]). A **Hom-Lie triple system** is a Hom-triple system  $(V, [, ], \alpha)$  satisfying the ternary Hom-Nambu identity (2.1).

When  $\alpha = Id$ , a Hom-Lie triple system reduces to a Lie triple system.

We now give the definition of the basic object of this paper.

**Definition 2.6.** A **Hom-Lie-Yamaguti algebra** (*Hom-LY algebra for short*) is a quadruple  $(L, *, \{, , \}, \alpha)$  in which  $L$  is a  $\mathbb{K}$ -vector space, “ $*$ ” a binary operation and “ $\{, , \}$ ” a ternary operation on  $L$ , and  $\alpha : L \rightarrow L$  a linear map such that

$$\begin{aligned}
 (\text{HLY1}) \quad & \alpha(x * y) = \alpha(x) * \alpha(y), \\
 (\text{HLY2}) \quad & \alpha(\{x, y, z\}) = \{\alpha(x), \alpha(y), \alpha(z)\}, \\
 (\text{HLY3}) \quad & x * y = -y * x, \\
 (\text{HLY4}) \quad & \{x, y, z\} = -\{y, x, z\}, \\
 (\text{HLY5}) \quad & \circlearrowleft_{(x,y,z)}[(x * y) * \alpha(z) + \{x, y, z\}] = 0, \\
 (\text{HLY6}) \quad & \circlearrowleft_{(x,y,z)}\{x * y, \alpha(z), \alpha(u)\} = 0, \\
 (\text{HLY7}) \quad & \{\alpha(x), \alpha(y), u * v\} = \{x, y, u\} * \alpha^2(v) + \alpha^2(u) * \{x, y, v\}, \\
 (\text{HLY8}) \quad & \{\alpha^2(x), \alpha^2(y), \{u, v, w\}\} = \{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\} \\
 & \quad \quad \quad + \{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\} \\
 & \quad \quad \quad + \{\alpha^2(u), \alpha^2(v), \{x, y, w\}\},
 \end{aligned}$$

for all  $u, v, w, x, y, z$  in  $L$ .

Note that the conditions (HLY1) and (HLY2) mean the multiplicativity of  $(L, *, \{, , \}, \alpha)$ .

**Remark.** (1) If  $\alpha = Id$ , then the Hom-LY algebra  $(L, *, \{, , \}, \alpha)$  reduces to a LY algebra  $(L, *, \{, , \})$  (see (LY1)-(LY6)).

(2) If  $x * y = 0$ , for all  $x, y \in L$ , then  $(L, *, \{, , \}, \alpha)$  becomes a Hom-Lie triple system  $(L, \{, , \}, \alpha^2)$  and, subsequently, a ternary Hom-Nambu algebra (since, by Definition 2.5, any Hom-Lie triple system is automatically a ternary Hom-Nambu algebra).

(3) If  $\{x, y, z\} = 0$  for all  $x, y, z \in L$ , then the Hom-LY algebra  $(L, *, \{, , \}, \alpha)$  becomes a Hom-Lie algebra  $(L, *, \alpha)$ .

### 3 Constructions of Hom-Lie-Yamaguti algebras

In this section we consider construction methods for Hom-LY algebras. These methods allow to find examples of Hom-LY algebras starting from ordinary LY algebras or even from Malcev algebras.

First, as the main tool, we show that the category of (multiplicative) Hom-LY algebras is closed under self-morphisms.

**Theorem 3.1.** *Let  $A_\alpha := (A, *, \{, , \}, \alpha)$  be a Hom-LY algebra and let  $\beta$  be an endomorphism of the algebra  $(A, *, \{, , \})$  such that  $\beta\alpha = \alpha\beta$ . Let  $\beta^0 = id$  and, for any  $n \geq 1$ ,  $\beta^n := \beta \circ \beta^{n-1}$ . Define on  $A$  the operations*

$$\begin{aligned} x *_\beta y &:= \beta^n(x * y), \\ \{x, y, z\}_\beta &:= \beta^{2n}(\{x, y, z\}) \end{aligned}$$

for all  $x, y, z$  in  $A$ . Then  $A_{\beta^n} := (A, *_\beta, \{, , \}_\beta, \beta^n\alpha)$  is a Hom-LY algebra, with  $n \geq 1$ .

**Proof.** First, we observe that the condition  $\beta\alpha = \alpha\beta$  implies  $\beta^n\alpha = \alpha\beta^n$ ,  $n \geq 1$ . Next we have

$$\begin{aligned} (\beta^n\alpha)(x *_\beta y) &= (\beta^n\alpha)(\beta^n(x) * \beta^n(y)) = \beta^n((\alpha\beta^n)(x) * (\alpha\beta^n)(y)) \\ &= (\alpha\beta^n)(x) *_\beta (\alpha\beta^n)(y) = (\beta^n\alpha)(x) *_\beta (\beta^n\alpha)(y) \end{aligned}$$

and we get (HLY1) for  $A_{\beta^n}$ . Likewise, the condition  $\beta\alpha = \alpha\beta$  implies (HLY2). The identities (HLY3) and (HLY4) for  $A_{\beta^n}$  follow from the skew-symmetry of “\*” and “{, , }” respectively.

Consider now  $\circlearrowleft_{(x,y,z)}((x *_\beta y) *_\beta (\beta^n\alpha)(z)) + \circlearrowleft_{(x,y,z)}\{x, y, z\}_\beta$ . Then

$$\begin{aligned} \circlearrowleft_{(x,y,z)}((x *_\beta y) *_\beta (\beta^n\alpha)(z)) + \circlearrowleft_{(x,y,z)}\{x, y, z\}_\beta &= \circlearrowleft_{(x,y,z)}[\beta^n(\beta^n(x * y) * \beta^n(\alpha(z)))] \\ &+ \circlearrowleft_{(x,y,z)}[\beta^{2n}(\{x, y, z\})] \\ &= \circlearrowleft_{(x,y,z)}[\beta^{2n}((x * y) * \alpha(z))] + \circlearrowleft_{(x,y,z)}[\beta^{2n}(\{x, y, z\})] \\ &= \beta^{2n}(\circlearrowleft_{(x,y,z)}[(x * y) * \alpha(z) + \{x, y, z\}]) \\ &= \beta(0) \text{ (by (HLY5) for } A_\alpha) \\ &= 0 \end{aligned}$$

and thus we get (HLY5) for  $A_{\beta^n}$ . Next,

$$\begin{aligned} \{x *_\beta y, (\beta^n\alpha)(z), (\beta^n\alpha)(u)\}_\beta &= \{\beta^{3n}(x * y), \beta^{3n}(\alpha(z)), \beta^{3n}(\alpha(u))\} \\ &= \beta^{3n}(\{x * y, \alpha(z), \alpha(u)\}). \end{aligned}$$

Therefore

$$\begin{aligned} \circlearrowleft_{(x,y,z)} \{x *_\beta y, (\beta^n\alpha)(z), (\beta^n\alpha)(u)\}_\beta &= \circlearrowleft_{(x,y,z)}[\beta^{3n}(\{x * y, \alpha(z), \alpha(u)\})] \\ &= \beta^{3n}(\circlearrowleft_{(x,y,z)}\{x * y, \alpha(z), \alpha(u)\}) \\ &= \beta^{3n}(0) \text{ (by (HLY6) for } A_\alpha) \\ &= 0 \end{aligned}$$

so that we get (HLY6) for  $A_{\beta^n}$ . Further, using (HLY7) for  $A_\alpha$  and condition  $\alpha\beta = \beta\alpha$ , we compute

$$\begin{aligned} \{(\beta^n\alpha)(x), (\beta^n\alpha)(y), u *_\beta v\}_\beta &= \beta^{3n}(\{\alpha(x), \alpha(y), u * v\}) \\ &= \beta^{3n}(\{x, y, u\} * \alpha^2(v) + \alpha^2(u) * \{x, y, v\}) = \beta^n(\beta^{2n}(\{x, y, u\}) * (\beta^{2n}\alpha^2)(v)) \end{aligned}$$



$$\begin{aligned}
 & +\beta^n((\beta^{2n}\alpha^2)(u) * \beta^{2n}(\{x, y, v\})) = \{x, y, u\}_\beta *_\beta (\beta^{2n}\alpha^2)(v) \\
 & +(\beta^{2n}\alpha^2)(u) *_\beta \{x, y, v\}_\beta \\
 & = \{x, y, u\}_\beta *_\beta (\beta^n\alpha)^2(v) + (\beta^n\alpha)^2(u) *_\beta \{x, y, v\}_\beta.
 \end{aligned}$$

Thus (HLY7) holds for  $A_{\beta^n}$ . Using repeatedly the condition  $\alpha\beta = \beta\alpha$  and the identity (HLY8) for  $A_\alpha$ , the verification of (HLY8) for  $A_{\beta^n}$  is as follows.

$$\begin{aligned}
 & \{(\beta^n\alpha)^2(x), (\beta^n\alpha)^2(y), \{u, v, w\}_\beta\}_\beta \\
 & = \{(\beta^{2n}\alpha^2)(x), (\beta^{2n}\alpha^2)(y), \{u, v, w\}_\beta\}_\beta \\
 & = \beta^{2n}(\{(\beta^{2n}\alpha^2)(x), (\beta^{2n}\alpha^2)(y), \beta^{2n}(\{u, v, w\})\}) \\
 & = \beta^{4n}(\{\alpha^2(x), \alpha^2(y), \{u, v, w\}\}) \\
 & = \beta^{4n}(\{\alpha^2(u), \alpha^2(v), \{x, y, w\}\}) \\
 & + \beta^{4n}(\{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\}) \\
 & + \beta^{4n}(\{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\}) \\
 & = \beta^{2n}(\{(\beta^{2n}\alpha^2)(u), (\beta^{2n}\alpha^2)(v), \beta^{2n}(\{x, y, w\})\}) \\
 & + \beta^{2n}(\{\beta^{2n}(\{x, y, u\}), (\beta^{2n}\alpha^2)(v), (\beta^{2n}\alpha^2)(w)\}) \\
 & + \beta^{2n}(\{(\beta^{2n}\alpha^2)(u), \beta^{2n}(\{x, y, v\}), (\beta^{2n}\alpha^2)(w)\}) \\
 & = \{(\beta^n\alpha)^2(u), (\beta^n\alpha)^2(v), \{x, y, w\}_\beta\}_\beta \\
 & + \{\{x, y, u\}_\beta, (\beta^n\alpha)^2(v), (\beta^n\alpha)^2(w)\}_\beta \\
 & + \{(\beta^n\alpha)^2(u), \{x, y, v\}_\beta, (\beta^n\alpha)^2(w)\}_\beta.
 \end{aligned}$$

Thus (HLY8) holds for  $A_{\beta^n}$ . Therefore, we get that  $A_{\beta^n}$  is a Hom-LY algebra. This finishes the proof. □

From Theorem 3.1 we have the following construction method of Hom-LY algebras from LY algebras (this yields examples of Hom-LY algebras). This method is an extension to binary-ternary algebras of a result due to D. Yau ([23], Theorem 2.3), giving a general construction method of Hom-algebras from their corresponding untwisted algebras. Such an extension to binary-ternary algebras is first mentioned in [6], Corollary 4.5.

**Corollary 3.2.** *Let  $(A, *, [, ,])$  be a LY algebra and  $\beta$  an endomorphism of  $(A, *, [, ,])$ . If define on  $A$  a binary operation " $\tilde{*}$ " and a ternary operation " $\{, , \}$ " by*

$$\begin{aligned}
 x\tilde{*}y & := \beta(x * y), \\
 \{x, y, z\} & := \beta^2([x, y, z]),
 \end{aligned}$$

*then  $(A, \tilde{*}, \{, , \}, \beta)$  is a Hom-LY algebra.*

**Proof.** The proof follows if observe that Corollary 3.2 is Theorem 3.1 when  $\alpha = Id$  and  $n = 1$ . □

A Malcev algebra [16] is an anticommutative algebra  $(A, *)$  such that the Malcev identity

$$J(x, y, x * z) = J(x, y, z) * x$$

holds for all  $x, y, z$  in  $A$ , where  $J(u, v, w) := \circlearrowleft_{(u,v,w)}(u * v) * w$  in  $(A, *)$ .

**Corollary 3.3.** Let  $(A, *)$  be a Malcev algebra and  $\beta$  any endomorphism of  $(A, *)$ . Define on  $A$  the operations

$$x \tilde{*} y := \beta(x * y),$$

$$\{x, y, z\} := \beta^2((x * y) * z - (y * z) * x - (z * x) * y).$$

Then  $(A, \tilde{*}, \{, \}, \beta)$  is a Hom-LY algebra.

**Proof.** If consider on  $A$  the ternary operation  $[x, y, z] := (x * y) * z - (y * z) * x - (z * x) * y, \forall x, y, z \in A$ , then  $(A, *, [, ,])$  is a LY algebra [20]. Moreover, since  $\beta$  is an endomorphism of  $(A, *)$ , we have  $\beta([x, y, z]) = (\beta(x) * \beta(y)) * \beta(z) - (\beta(y) * \beta(z)) * \beta(x) - (\beta(z) * \beta(x)) * \beta(y) = [\beta(x), \beta(y), \beta(z)]$  so that  $\beta$  is also an endomorphism of  $(A, *, [, ,])$ . Then Corollary 3.2 implies that  $(A, \tilde{*}, \{, \}, \beta)$  is a Hom-LY algebra. □

## 4 Examples of 2-dimensional Hom-Lie-Yamaguti algebras

In this section, algebras are considered over the ground field of real numbers. Using the classification of real 2-dimensional LY algebras ([5], Corollary 2.7), we classify all the algebra morphisms of each of the proper (i.e. nontrivial) 2-dimensional LY algebras and then we obtain their corresponding 2-dimensional Hom-LY algebras (hence we give some application of Corollary 3.2).

For completeness, we recall that any 2-dimensional real LY algebra is isomorphic to one of the algebras (with basis  $\{u, v\}$ ) of the following types:

(T1)  $u * v = 0, [u, v, u] = \lambda u + \mu v, [u, v, v] = \gamma u - \lambda v;$

(T2)  $u * v = u, [u, v, u] = 0, [u, v, v] = ku;$

(T3)  $u * v = u + v, [u, v, u] = 0, [u, v, v] = 0;$

(T4)  $u * v = au + bv, [u, v, u] = eu + fv, [u, v, v] = ku - ev$

with  $a \neq 0$ ,  $b \neq 0$ ,  $e \neq 0$ ,  $f \neq 0$ ,  $k \neq 0$ , and  $af - be = 0 = bk + ae$ . Also recall that a classification of complex 2-dimensional LY algebras is given in [21].

The algebras of type (T1) are either the zero algebra or nonzero Lie triple systems (a classification of real 2-dimensional Lie triple systems is given in [10]; see also [3], a classification of complex 2-dimensional Lie triple systems is found in [19]). According to Corollary 3.6 in [25], each of these Lie triple systems with a given endomorphism gives rise to a (real 2-dimensional) Hom-Lie triple system (i.e. a Hom-LY algebra with zero binary operation).

The algebras of type (T3) are real nonzero 2-dimensional Lie algebras (in fact (T3) is the only one, up to isomorphism, real nonzero 2-dimensional Lie algebra). Given any endomorphism of (T3) we get, by Theorem 3.3 of [23], its corresponding (real 2-dimensional) Hom-Lie algebra (i.e. a Hom-LY algebra with zero ternary operation).

Since Hom-Lie algebras as well as Hom-Lie triple systems are particular instances of Hom-LY algebras, we shall focus on algebras of types (T2) and (T4) in order to get nontrivial applications of Corollary 3.2.

In the LY algebra of type (T4), consider the basis change  $\tilde{u} = au + bv$ ,  $\tilde{v} = v$ . Then some few transformations and the use of conditions  $a \neq 0$ , and  $af - be = 0 = bk + ae$  imply that (T4) is isomorphic to

$$(T4') \quad \tilde{u} * \tilde{v} = a\tilde{u}, [\tilde{u}, \tilde{v}, \tilde{u}] = 0, [\tilde{u}, \tilde{v}, \tilde{v}] = k\tilde{u}.$$

One observes that the algebra (T4') is isomorphic to the algebra (T2) by an isomorphism  $\theta$  given by  $\theta(u) = \alpha\tilde{u} + \beta\tilde{v}$ ,  $\theta(v) = \gamma\tilde{u} + \delta\tilde{v}$  if and only if  $a = \pm 1$  and  $\alpha \neq 0$ . Then the isomorphisms are

$$\theta(u) = \alpha\tilde{u}, \theta(v) = \gamma\tilde{u} + v \text{ (when } a = 1)$$

and

$$\theta(u) = \alpha\tilde{u}, \theta(v) = \gamma\tilde{u} - v \text{ (when } a = -1),$$

where  $\alpha \neq 0$  and  $\gamma$  are real numbers.

Therefore, for a nontrivial application of Corollary 3.2, we shall consider algebras of type (T2) and type (T4') when  $a \neq \pm 1$ .

In general, if  $(A, *, [, ,])$  is a LY algebra, then a linear self-map  $\phi$  of  $A$  is an endomorphism of  $(A, *, [, ,])$  if

$$\begin{cases} \phi(x * y) = \phi(x) * \phi(y) \\ \phi([x, y, z]) = [\phi(x), \phi(y), \phi(z)], \end{cases} \quad (4.1)$$

for all  $x, y, z \in A$ . In order to determine  $\phi$ , it suffices to apply the conditions (4.1) to the basis elements of  $A$ . Using elementary algebra, we are led to

solve the resulting  $n$  simultaneous equations with respect to the coefficients expressing  $\phi$  in the given basis (the number  $n$  depends on the dimension of  $A$ ). These equations are not difficult to solve when  $\dim A = 2$ . Suppose now that  $\dim A = 2$  and let  $\{u, v\}$  be a basis of  $A$ . Then a linear self-map  $\phi$  of  $A$  is given by a  $2 \times 2$ -matrix with respect to  $\{u, v\}$ . Set  $\phi(u) = \epsilon u + \beta v$ ,  $\phi(v) = \gamma u + \delta v$ .

★ Case of type (T2):

Then (4.1) induces the following simultaneous equations:

$$\begin{cases} \beta = 0 \\ \epsilon(1 - \delta) = 0, \end{cases}$$

$$\begin{cases} \beta = 0 \\ \epsilon(1 - \delta^2) = 0. \end{cases}$$

Resolving these simultaneous equations, we are led to the following endomorphisms  $\phi$  of (T2):

$$I. \begin{cases} \phi(u) = 0 \\ \phi(v) = \gamma u + \delta v, \end{cases}$$

$$II. \begin{cases} \phi(u) = \epsilon u \\ \phi(v) = \gamma u + v, \end{cases}$$

where  $\gamma, \delta$ , and  $\epsilon \neq 0$  are any real numbers.

For each endomorphism I. and II. above, we apply Corollary 3.2 to find the corresponding Hom-LY algebra.

-For an endomorphism of type I., Corollary 3.2 implies that the LY algebra (T2) is twisted into the zero Hom-LY algebra by such an endomorphism.

-For an endomorphism of type II., Corollary 3.2 induces from (T2) the following Hom-LY algebras:

$$u\tilde{*}v = \epsilon u, \{u, v, u\} = 0, \{u, v, v\} = k\epsilon^2 u, \tag{4.2}$$

where  $\epsilon \neq 0$  and  $k \neq 0$  are real numbers.

**Remark:** A LY algebra of type (T2) admits a nonzero twisting (i.e. the associated Hom-LY algebra is nonzero) if and only if  $\epsilon \neq 0$ , in which case the twisting maps are algebra automorphisms (including the identity map) of (T2).

★ Case of type (T4') (i.e. (T4)):

In this case, (4.1) leads to the same types I. and II. of endomorphisms as above. Therefore:

-for an endomorphism of type I., we get the zero Hom-LY algebra from (T4') by Corollary 3.2;

-for an endomorphism of type II., the application of Corollary 3.2 to (T4') gives the following Hom-LY algebras:

$$\tilde{u}\tilde{*}\tilde{v} = a\epsilon\tilde{u}, \{\tilde{u}, \tilde{v}, \tilde{u}\} = 0, \{\tilde{u}, \tilde{v}, \tilde{v}\} = k\epsilon^2\tilde{u}, \quad (4.3)$$

where  $\epsilon \neq 0$ ,  $a \neq 0, \pm 1$  and  $k \neq 0$  are real numbers.

**Remark:** One observes that a Hom-LY algebra of type (4.2) is isomorphic to the one of type (4.3) by an isomorphism  $\theta(u) = \alpha\tilde{u} + \beta\tilde{v}$ ,  $\theta(v) = \gamma\tilde{u} + \delta\tilde{v}$  if and only if  $a = \pm 1$  and  $\alpha \neq 0$ . Then the isomorphisms are

$$\theta(u) = \alpha\tilde{u}, \theta(v) = \gamma\tilde{u} + \tilde{v} \text{ (when } a = 1)$$

and

$$\theta(u) = \alpha\tilde{u}, \theta(v) = \gamma\tilde{u} - \tilde{v} \text{ (when } a = -1).$$

Therefore the Hom-LY algebras (4.2) and (4.3) are isomorphic if and only if the LY algebras (T2) and (T4') are isomorphic.

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